Semantic Closure and Classicality

Cierre semántico y clasicalidad

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Abstract
The semantic paradoxes show that semantic theories that internalize their semantic concepts, such as truth and validity, cannot validate all classical logic. That is, it is necessary to weaken some connective of the object language, taken as the guilty of the paradoxes, or give up some property of the consequence relation of the logical theory. Both strategies may distance us from classical logic, the logic commonly used in our current mathematical theories. So, a desirable solution to semantic paradoxes cannot distance us from classical logic. This paper analyzes two interesting proposals that aim to maintain classical logic as much as possible. The first strategy, the Barrio & Pailos & Szmuc-approach (2017) (BPS-approach), proposes the paraconsistent logic MSC that contains in its object language a connective capable of recovering classical inference whenever the sentences at issue are consistent. So, they show that it is possible to build a semantic theory over this logic that is immune to the semantic paradoxes. The second approach is based on the hierarchy ST_{\omega} of non-transitive systems ST_n, proposed by Pailos (2020a). This hierarchy recovers classical metainferences as many as possible in higher levels of the hierarchy. We argue in favor of the second approach by arguing that the first strategy must adopt weak self-referential procedures to avoid the paradoxes.

Keywords: semantic paradoxes, recovery operators, paraconsistent logics, substructural logics, classical logic.

Resumen
Las paradojas semánticas muestran que las teorías semánticas que internalizan sus propios conceptos semánticos, como la verdad y la validez, no pueden validar toda la lógica clásica. Es decir, es necesario debilitar algún conectivo del lenguaje objeto, tomado como culpable
1. Introduction

The semantic paradoxes impose a limit in the project of formulating a semantic theory \( T \) capable of expressing its own semantic concepts. First, by Tarski’s undefinability theorem (Tarski, 1956; 1933), it is impossible to introduce the truth predicate that states “\([A]\) is true if and only if \(A\)”, where \([A]\) is the name of \(A\), in the object language of semantic theories \( T \) based on classical logic that are able to express their own syntax, due to the inconsistency caused by Truth paradox. Second, as Beall & Murzi (2013), these theories are not capable to express their on validity predicate that states “there is a valid derivation of \([B]\) from \([A]\)” due to the inconsistency caused by Curry paradox. These impossibilities results are due to the expressive capability of \(T\) of formulating self-referential statements.

As it is widely known, many solutions to these paradoxes propose to weaken the base logic \(L\) of \(T\) by weakening the deductive behaviour of the logical connectives (Kripke, 1976; Goodship, 1996; Priest, 2006; Field, 2008; Barrio et al., 2017; Pailos, 2020b) or by weakening the deductive properties of the deductive relation of \(L\) (Ripley, 2013; Zardini, 2013; Meadows, 2014; Weber, 2014; French, 2016; Murzi and Rossi, 2021). Each solution blocks these paradoxes in a specific way.

On the other hand, in blocking these inconsistency results, many of these solutions pay a high cost. By dropping one or more properties of classical logic, these non-classical solutions fail to validate many inferences that are usually performed in the current scientific practice. This preoccupation with the recapture of classical inferences is not new in the literature. Indeed, the forefather of paraconsistency Da Costa (1974) introduced his hierarchy of
paraconsistent systems that are able to recover classical inferences with the aid of classicality connectives. Although his works on paraconsistency did not have the preoccupation with semantic paradoxes, it shows that the preoccupation with classical recapture is not a novelty. Priest (2006) introduces the minimally inconsistent logic mLP in order to recover classical inferences when there is no paradoxical sentence involved.

The classical recapture was vastly explored in recent years. Da Costa’s original idea deeply influenced the Logics of Formal Inconsistency (Carnielli et al. 2007; Carnielli et al., 2016). The Logics of Formal Inconsistency (henceforth, LFIs) are a family of paraconsistent systems that recovery classical inferences once some consistency assumptions are made. These logics have classicality connectives that makes possible the recovery of classical inferences. This interesting idea was widely discussed and applied to distinct contexts. Antunes (2020) and Tajer (2020) show that these logics have sufficient expressive power to accommodate Priest’s recapture proposal as well as Beall’s shrieking method (Beall, 2013).

However, this recovery strategy does not work well in the context of semantic paradoxes. As Barrio et al. (2017) proves, many theories of truth that have the expressive strenght of LFIs are trivial. So it is also necessary to weaken the self-referential procedures of these semantic theories with the aid of weak biconditionals. But, as Rosenblatt (2021) argues, this movement is ad hoc, because there is no clear justification of why it is necessary to weaken the self-referential procedures. Moreover, as Picollo (2020) shows, these classicality connectives fail in recovering the validity of instances of the induction axioms in some non-classical theories of truth. Then, as these criticisms shows, the introduction of recovery operators face significant difficulties in the context of semantic theories.

In this paper, we argue that the semantic theory STT$\omega$, introduced by Pailos (2020a) and explored in Barrio et al. (2021), offers an adequate approach to a semantic theory of truth, and can be non-trivially extended with a validity predicate. STT$\omega$ is comprised by an infinite hierarchy of non-transitive systems that allows to recover the classical metainferences we lose at each step of the hierarchy. This paper is organized as follows. In the Section 2, we present the paradoxes of truth and validity. We also discuss the problems concerning the deductive weakness of the non-classical solutions to these paradoxes. In Section the 3, we present Barrio et al. (2017)’s proposal, which proposes to introduce a consistency connective in paraconsistent theories of truth in order to recover classical inferences whenever the sentences involved are not paradoxical. We also present some criticisms that this approach faces. In the Section 4, we present the hierarchy of strict tolerant logics, introduced by Pailos (2020a), that can be a basis to a non-trivial theory of truth. In the Subsection 4.1, we discuss the advantages of this approach, some criticisms that this proposal faces, and we respond these criticisms. In the Section 5, we close the discussion with a few remarks.
2. Paradoxes of truth and validity

Tarski’s undefinability theorem shows that no theory $T$ based on classical logic and that has expressive power to talk about its own syntax can have a truth predicate $T_r$ that satisfies the following schema:

(Schema-$Tr$) \[ Tr([A]) \leftrightarrow A. \]

Where $[A]$ is the name of $A$. The reason for such impossibility is that these theories have sufficient deductive power to prove Diagonalization lemma, which allow them self-referential statements and, among them, the sentence that state its own falsity:

(Liar sentence) \[ \lambda \leftrightarrow \neg Tr([\lambda]). \]

Let $|=_T$ be the consequence relation of $T$. The (Liar sentence) generate the inconsistency known as truth paradox, whose derivation is given as follows:

1. $|=_{T} \lambda \leftrightarrow \neg Tr([\lambda])$ (Liar sentence)
2. $|=_{T} \lambda \leftrightarrow Tr([\lambda])$ (Schema-Tr)
3. $|=_{T} Tr([\lambda]) \leftrightarrow \neg Tr([\lambda])$ Transitivity of $\leftrightarrow 1, 2$
4. $|=_{T} Tr([\lambda]) \land \neg Tr([\lambda])$ Elimination of $\leftrightarrow, 3$
5. $|=_{T} (Tr([\lambda]) \land \neg Tr([\lambda])) \rightarrow \bot$ Explosion
6. $|=_{T} \bot$ Modus Ponens 4, 5

It is also possible to formulate the sentence that says an absurdity follows from its own truth:

(Curry sentence) \[ c \leftrightarrow (Tr([c]) \rightarrow \bot) \]

The (Curry sentence) generate the Curry paradox, whose derivation is given as follows:

1. $|=_{T} c \leftrightarrow (Tr([c]) \rightarrow \bot)$ (Curry sentence)
2. $|=_{T} c \leftrightarrow Tr([c])$ (Schema-Tr)
3. $|=_{T} Tr([c]) \leftrightarrow (Tr([c]) \rightarrow \bot)$ Transitivity of $\leftrightarrow 1, 2$
4. $|=_{T} Tr([c]) \rightarrow (Tr([c]) \rightarrow \bot)$ Elimination of $\leftrightarrow 3$
5. $|=_{T} Tr([c]) \rightarrow \bot$ Contraction 4
6. $|=_{T} (Tr([c]) \rightarrow \bot) \rightarrow c$ Elimination of $\leftrightarrow 1$
7. $|=_{T} c$ Modus Ponens 5, 6
8. $|=_{T} c \rightarrow Tr([c])$ Elimination of $\leftrightarrow 2$
9. $|=_{T} Tr([c])$ Modus Ponens 7, 8
10. $|=_{T} \bot$ Modus Ponens 5, 9
So, no theory based on classical logic that has expressive power to talk about its own syntax can have such a truth predicate $Tr$. As we know, Tarski original solution appeal to a transfinite hierarchy of metalanguages where the truth predicate relative to the language $L$ is only definable in its metalanguage.

On the other hand, the basic idea of non-classical solutions to both paradoxes is to maintain the truth predicate in the object language $L$ of $T$ by changing the underlying logic $L$. So, as Beall & Murzi (2013) observe, the solutions that serve to block the truth paradox by weakening the negation, dropping Explosion (in symbols $A, \neg A \models B$) or Excluded Middle ($\models A \vee \neg A$), are not sufficient. It is also necessary to weaken the connective $\rightarrow$. So, most paraconsistent logics – i.e. logics that drop Explosion- and paracomplete logics – i.e., logics that drop Excluded Middle- are not adequate to give a uniform solution to both paradoxes. Among the paraconsistent solutions to truth paradoxes, we find Goodship (1996), Priest (2006), Barrio et al. (2017) and Pailos (2020). Among the paracomplete ones, we find Kripke (1976) and Field, 2008.\(^1\)

The situation become even worse when one wants to internalize other logical concepts into the object language of the logic. For example, consider the binary validity predicate $Val$, that internalizes the following relation between sentences: there is a valid derivation of $B$ from $A$. This predicate is supposed to satisfy the following rules:

(Validity Proof) Given a valid derivation of $B$ from $A$, infer $Val([A],[B])$.

(Validity Detachment) From $A$ and $Val([A],[B])$ infer $B$.

The rule (Validity Proof), henceforth (VP), says that the valid inferences of $T$ are also valid with respect of the predicate $Val$. The rule (Validity Detachment) says that truth is preserved under Modus Ponens. According to Beall & Murzi, this predicate, when introduced in the object language of $T$, has catastrophic effects. The main reason is that its derivation uses mainly the structural rules of the consequence relation $\models_T$ of $T$, such as:

(Reflexivity) $\Gamma, A \models_T A$;

(Contraction) If $\Gamma, A, A \models_T B$, then $\Gamma, A \models_T B$.

(Cut) If $\Gamma \models_T B$ and $\Delta, B \models_T C$, then $\Gamma, \Delta \models_T C$.

Given that the $T$ has expressive power to generate self-referential statements, it is possible to formulate the following statement:

(Validity Curry) $\epsilon \leftrightarrow Val([\epsilon],[\bot])$

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\(^1\) This list is not meant to be exhaustive. We refer the reader to Barrio (2014) for a good presentation of the non-classical solutions to truth paradoxes.
The sentence (Validity Curry) generates the validity paradox, a.k.a. \( \nu \)-Curry, whose derivation is given as follows:

1. \( \models_T c \leftrightarrow \text{Val}([c],[\bot]) \) (Validity Curry)
2. \( c \models_T c \) (Reflexivity)
3. \( \models_T c \rightarrow \text{Val}([c],[\bot]) \) (Elimination of \( \leftrightarrow \), 1)
4. \( c \models_T \text{Val}([c],[\bot]) \) (Modus Ponens 2, 3)
5. \( c, c \models_T \bot \) (VD) 2, 4
6. \( \models_T \bot \) (Contraction) 5
7. \( \models_T \text{Val}([c],[\bot]) \) (VP) 6
8. \( \models_T \text{Val}([c],[\bot]) \rightarrow c \) (Elimination of \( \leftrightarrow \), 1)
9. \( \models_T c \) (Modus Ponens 7, 8)
10. \( \models_T \bot \) (VD) 7, 9

Although (Transitivity) was not explicitly used in the derivation, it is hidden in a formulation of the rule (VD), which can be formulated in the following two ways:

\[(\text{VD}_1) A, \text{Val}([A],[B]) \models_T B.\]

\[(\text{VD}_2) \models_T A \text{ and } \models_T \text{Val}([A],[B]) \text{ imply } \models_T B.\]

The rule (VD) is used in the derivation of \( \nu \)-Curry. As we will show now, this rule can be proven from (VD) and (Transitivity).

1. \( \models_T A \) Hypothesis
2. \( \models_T \text{Val}([A],[B]) \) Hypothesis
3. \( A, \text{Val}([A],[B]) \models_T B \) (VD)
4. \( \text{Val}([A],[B]) \models_T B \) (Transitivity) 1, 3
5. \( \models_T B \) (Transitivity) 2, 4

The distinction between (VD) and (VD) is important, because the non-classical solutions that dispense the property (Transitivity) cannot introduce the rule (VD) given it presupposes (Transitivity).

As the derivation of \( \nu \)-Curry shows, restrictions on the connectives may not be sufficient, given that the above derivation mainly uses the rules of the predicate \text{Val} as well as the properties of \( \models_T \). Some non-classical solutions proceed by dispensing one or more properties of \( \models_T \). These solutions are called \textit{substructural solutions}. For example, French (2016), Murzi...
& Rossi (2017) propose theories of validity and truth that dispense (Reflexivity); Zardini (2013) and Weber (2014) propose theories that dispense (Contraction); and Ripley (2013) and Barrio et al. (2016) propose theories that dispense (Transitivity).

Although v-Curry mainly uses the properties of the consequence relation of $\mathcal{T}$, it is possible to present theories of validity and truth that keep all these properties of $|=_{\mathcal{T}}$ intact. For example, Pailos (2020b) proposes a paraconsistent theory of validity and truth that respects the three properties of $|=_{\mathcal{T}}$. But, in order to avoid v-Curry, he adopts a weaker self-referential procedure in his theory.

As widely discussed in the literature about semantic paradoxes, many non-classical solutions fail in delivering a theory that is strong enough to validate some inferences that are usually performed in our inferential practice. For example, if a semantic theory fails in validating the rule of Modus Ponens, one is entitled to say that this theory fail in delivering a good theory of reasoning about these semantic concepts. The same could be said with respect to a theory that does not have any valid formula. So, the closeness to the classical reasoning is still a good way for adopting a non-classical solution to paradoxes. In what follows, we will discuss two possible approaches to classicality. One of them is proposed by Barrio, Pailos and Szmuc (2017), the BPS approach, and the other is the hierarchy of non-transitive systems proposed by Pailos (2020a) and Barrio et al. (2021). Although both approaches have their merits and own internal problems, we will argue in favour of the latter approach.

Although the validity paradoxes play an important role in the debate, we will focus on the theories of truth. The main reason is that our discussion is more focused on the classicality of the theories, and that solutions below are also able to block the validities paradoxes when properly extended.

3. Classicality and paradoxes

As we discussed in the latter Section, many non-classical solutions sacrifice their deductive power in order to introduce semantic notions in their object language. So, it is desirable that some method of classical recapture should be available whenever problematic sentences such as (Liar sentence) are not involved. In the literature about classical recapture, we find many strategies such as Priest’s logic mLP (Priest, 2006) Beall’s shrieking method (Beall, 2013). Both strategies show that it is possible to recover classical inferences whenever we are facing neither sentences that are both true and false nor sentences that are neither true nor false. On the other hand, as Antunes (2020) and Tajer (2020) argue, the strategy of introducing recovery operators in the object language is able to reproduce these strategies in a more elegant way.

The strategy of introducing recovery operators in the object language of formal theories is not new in the literature. Indeed, Da Costa (1974) introduced these operators that separate the well-behaved sentences from the sentences that are not. This idea was widely explored by
the founders of the *Logics of Formal Inconsistency* (LFIs) (Carnielli *et al*., 2007; Carnielli *et al*., 2016). LFIs are paraconsistent logics that, although they do not validate the inference $A, \neg A \models B$, they have a consistency connective $\circ$ such that:

- (i) $\circ A, A \not\models B$ ($\not\models$ means that the relation $\models$ does not hold)
- (ii) $\circ A, \neg A \not\models B$
- (iii) $\circ A, A, \neg A \models B$

This last condition is called *gentle principle of explosion*. Intuitively, it says that if $A$ is consistent and both $A$ and $\neg A$ are true, then anything follows. LFIs comprise a wide family of paraconsistent logics that have $\circ$ as primitive or that is capable to define $\circ$ by means of other primitive connectives of the language. Although we will focus on these logics, it is important to mention the existence of the *Logics of Formal Undeterminedness* (LFUs) (Marcos, 2005). LFUs are logics that, although they do not validate $\models A \lor \neg A$, they have a determinedness connective $\$ such that:

$\$A \models A \lor \neg A$

BPS propose a semantic theory of truth based on a LFI. The logical system they propose is called MSC (*Matrix Logic for Semantic Closure*). This logic has the language $L_{MSC} = \{\text{Var}, \neg, \circ, \land, \lor, \rightarrow_{MSC}, \leftrightarrow_{MSC}\}$. Its set of formulas $For_{MSC}$ is generated as usual. The semantic structure is the matrix $M_{MSC} = (\{1, \frac{1}{2}, 0\}, \neg, \circ, \land, \lor, \rightarrow_{MSC}, \leftrightarrow_{MSC}, \{1, \frac{1}{2}\})$, where $\{1, \frac{1}{2}, 0\}$ is the set of truth values, $\{1, \frac{1}{2}\}$ is the set of designated values, $\neg, \circ, \land, \lor, \rightarrow_{MSC}, \leftrightarrow_{MSC}$ are operations that have the following truth-tables:

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Let $v : \text{For}_{\text{MSC}} \rightarrow \{1, \frac{1}{2}, 0\}$ be a valuation and $\text{SEM}_{\text{MSC}}$ be the set of all valuations $v$. We say that $v$ is a model of $A \in \text{For}_{\text{MSC}}$ if $v(A) \in \{1, \frac{1}{2}, 0\}$. $A$ is a tautology if every valuation $v$ is a model of $A$. $A$ is a semantic consequence of a set of formulas $\Gamma$ ($\Gamma \models_{\text{MSC}} A$) if and only if: if $v$ is a model of every $\gamma$ of $\Gamma$, then $v$ is also a model of $A$.

Given the truth tables of the negation and the consistency connective, it is easy to see that MSC is a LFI. As the following proposition shows, one can easily show that $\rightarrow_{\text{MSC}}$ does not validate Modus Ponens.

**Proposition A:** $A, A \rightarrow_{\text{MSC}} B \models / B$.

**Proof:** Let $v \in \text{SEM}_{\text{MSC}}$ be a valuation such that $v(A) = \frac{1}{2}$ and $v(B) = 0$. By the semantic definition of $\rightarrow_{\text{MSC}}$, $v(A \rightarrow_{\text{MSC}} B) = \frac{1}{2}$. Then, $A, A \rightarrow_{\text{MSC}} B \models / B$. This concludes the proof. Q.E.D.

As said before, MSC is a LFI. With the aid of the connective $\circ$ it is possible to recover classical inferences. This is possible due to Derivability Adjustment Theorem (DAT). According to this Theorem, classical inferences hold if the formulas involved are consistent. One of the first formulation of this result in the literature is given by Da Costa (1974) for his systems Co. There are many versions of this result in the literature. For example, Carnielli et al. (2020) provide a wide investigation of these connectives in the field of LFIs and LFUs, and Ciuni & Carrara (2020) investigate these connectives in many-valued logics.

Let CPL be the classical propositional logic. The following Theorem is a version of DAT for MSC. Its proof follows from Ciuni & Carrara’s results:

**Theorem B.** For every subset $\Gamma$ of $\text{For}_{\text{MSC}}$ and for every $A$ of $\text{For}_{\text{MSC}}$,

$$\Gamma \models_{\text{CPL}} A \text{ if and only if } \{o_p, \ldots, o_p\} \Gamma \models_{\text{MSC}} A,$$

where $\{p_1, \ldots, p_n\} \in \mathbb{N}$ is the set of propositional variables that occur in $\Gamma \cup \{A\}$.

Thus, Theorem B allows to recover classical inferences whenever the sentences at issue are not the problematic ones, such as (Liar sentence) and (Curry sentence). Now, consider a semantic theory of truth $T_{\text{MSC}}$ whose logical basis is MSC. $T_{\text{MSC}}$ is supposed to be strong enough to provide names $[A]$ for its sentences $A$. In this theory, the truth predicate is semantically characterized as:

$$v(A) = v(\text{Tr}([A]))$$

Unfortunately, the strong expressiveness of $T_{\text{MSC}}$ is a problem. As Tajer (2020) observes, recovery operators and self-reference do not match well. The main reason is the formulation of the following sentence:

$$A \leftrightarrow (oA \land \neg \text{Tr}([A]))$$

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2 We also refer the reader to Corbalán (2012)’s master dissertation for a deep study of these connectives.
As BPS observe, the presence of (Stronger liar) makes many semantic theories trivial. In order to avoid triviality, they adopt a weak self-referential principle for $T_{MSC}$, which stated as follows:

**Weak Self-Referential Principle (WSRP).** Let $T$ be a theory that has a name forming device $[.]$. If for every formula $\psi(x)$, with $x$ as the only free variable in $\psi(x)$, there is a sentence $\phi$ such that the formula $\phi \leftrightarrow \psi([\phi])$ is true in $T$, then we say that $T$ adopts WSRP.

In the case of $T_{MSC}$, the biconditional of WSRP is $\leftrightarrow_{MSC}$. By the semantic conditions of this connective, if we assign $\frac{1}{2}$ to the left-hand side formula, the whole biconditional will receive $\frac{1}{2}$, disregarding the truth-value of the right-hand side formula. So, the non triviality of $T_{MSC}$ is guaranteed when $T_{MSC}$ is extended with WSRP, where the biconditional in question is $\leftrightarrow_{MSC}$. As BPS observe, the adoption of any stronger self-referential principle leads $T_{MSC}$ to triviality. For example, if we allow a self-referential principle be stated in terms of ‘=’ instead of the biconditional $\leftrightarrow_{MSC}$, $T_{MSC}$ becomes trivial.

However, BPS’ approach face some objections. Rosenblatt (2021) argues that BPS’ proposal is ad hoc. As he argues, it is possible to extend the language of $T_{MSC}$ with primitive recursive functions that make it possible to prove a stronger self-referential principle. But, as we said before, this makes $T_{MSC}$ trivial. If non-triviality is the only reason to introduce WSRP instead of any stronger self-referential principle, BPS’s proposal is philosophically unsatisfactory.

Another criticism comes from Picollo (2020), who argues that theories of truth based on LFIs fail in validating important arithmetical axioms, such as the induction axiom. In her work, she focuses on the truth theory based on the logic $LP_o$, that has the same truth-tables of conjunction, negation and consistency connective as MSC. Its implication connective also fails to validate Modus Ponens. So, the connective $o$ fails in its basic aim: recovering the validity of classical inferences. As she argues, our modifications in the logical theories may affect the way we do mathematics.

Since LFIs fail in capturing classical inferences in the contexts of stronger theories of truth, the ones that extend theories $T$ in the arithmetical language, one might say that BPS approach has a narrow scope. In what follows, we argue that the theory of truth based on the metainferential logic $ST_\omega$ does better at recovering classical inferences. As we will see, the logical base $ST$ has many merits, by sharing the same set of tautologies and inferences as classical logic, differing on the level of metainferences (i.e., inferences between inferences). But, as Pailos (2020a), Barrio et al. (2020) and Barrio et al. (2021) shows, the metainferences that $ST$ fails to validate are recovered in the logic $ST_1$, thus creating the hierarchy $ST_\omega$ where the logics in the higher steps recover the metainferences of the logics of the lower levels. We also respond some objections made against the hierarchy $ST_\omega$. 

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4. The theory of truth STT_ω

In this Section we introduce the theory of truth STT_ω. First, we will introduce some basic definitions. We also respond some possible criticism against STT_ω. We will present now the Strict Tolerant Logic (ST) (Cobreros et al., 2012; Ripley, 2012, 2013). The logic ST is characterized by the matrix M_{ST} = \{\{1, \frac{1}{2}, 0\}, \sim, \land, \lor,\{1\}, \{1, \frac{1}{2}\}\}, where \{1, \frac{1}{2}, 0\} is the set of truth-values, \sim, \land, \lor are operations that share the same truth tables as the truth tables of the logic MSC, \{1\} is the set of strict truth, and \{1, \frac{1}{2}\} is the set of tolerant truth.

Let \( v : \text{For}_{ST} \to \{1, \frac{1}{2}, 0\} \) be a valuation and \( \text{SEM}_{ST} \) be the set of all valuations \( v \). We say that \( v \) is a model of \( A \in \text{For}_{ST} \) if \( v(A) \in \{1, \frac{1}{2}, 0\} \). A is a tautology if every valuation \( v \) is a model of \( A \). A is a semantic consequence of a set of formulas \( \Gamma \) (\( \Gamma \models_{ST} A \)) if and only if:

\[ \text{if } v(\gamma) = 1, \text{ for every } \gamma \in \Gamma, \text{then } v(A) \in \{1, \frac{1}{2}\} \]

Given the definition of \( \models_{ST} \), one can show that it is non-transitive. Suppose that \( \Gamma = \Delta = \emptyset \), \( v(A) = 1 \), \( v(B) = \frac{1}{2} \) and \( v(C) = 0 \) for every \( v \) of \( \text{SEM}_{ST} \). By definition of \( \models_{ST} A \models_{ST} B \) and \( B \models_{ST} C \), but \( A \models_{ST} C \). For being cut-free, this logic can be a non-trivial basis for semantic theories of truth.

From the truth tables of \( \sim, \land, \lor, \), one may define the semantic structure \( M_{TS} = \{\{1, \frac{1}{2}, 0\}, \sim, \land, \lor, \{1\}, \{1, \frac{1}{2}\}, \{1\}\} \) for the Tolerant Strict Logic (TS) (French, 2016). This logic will be important for our objectives in this Section. Let \( v : \text{For}_{TS} \to \{1, \frac{1}{2}, 0\} \) be a valuation and \( \text{SEM}_{TS} \) be the set of all valuations \( v \). We say that \( v \) is a model of \( A \in \text{For}_{TS} \) if \( v(A) = 1 \). A is a tautology if every valuation \( v \) is a model of \( A \). A is a semantic consequence of a set of formulas \( \Gamma \) (\( \Gamma \models_{TS} A \)) if and only if:

\[ \text{if } v(\gamma) \in \{1, \frac{1}{2}\}, \text{ for every } \gamma \in \Gamma, \text{then } v(A) = 1. \]

The logic TS is non-reflexive. Let \( v : \text{For}_{TS} \to \{1, \frac{1}{2}, 0\} \) be a valuation of \( \text{SEM}_{TS} \) such that \( v(A) = \frac{1}{2} \). By definition of \( \models_{TS} \), it is clear that \( A \models_{TS} A \). This logic can also be used as a basis for a semantic theory of truth given that it blocks the rule of reflexivity.

In (Ripley, 2013), Ripley propose a theory of truth, which we call here STT, that is obtained by extending the language of ST with the predicate Tr as well as names for the sentences of ST. Given that ST is cut-free, STT is immune to the paradoxes of truth and validity. An interesting advantage of ST over the others non-classical solutions is its closeness to classical logic. As Ripley shows, ST has the same tautologies and valid inferences as classical logic. However, Ripley’s proposal faces some criticisms. The main criticism his theory faces is that it is not closed under its own validities. For example, although ST validates explosion, it validates neither the metarule of explosion nor the metarule of modus ponens:

\[ \text{(Meta-explosion) } \models_{ST} A \text{ and } \models_{ST} \sim A \text{ imply } \models_{ST} B. \]

\[ \text{(Meta-modus ponens) } \models_{ST} A \text{ and } \models_{ST} A \rightarrow B \text{ imply } \models_{ST} B. \]
Because of such a failure, some truth-theorists claim that it is not possible to reason using this logic because it is not closed under its own valid inferences. But, there are proposals in the literature that show that ST can be extended in such a way that classical metainferences (inferences between inferences) are always recovered. In the last few years, the Buenos Aires Logic group published a series of papers where they propose an infinite hierarchy of metainferential logics based on ST (Pailos, 2020a; Barrio et al., 2018; Barrio et al., 2019; Barrio et al., 2021). The intuition behind this interesting idea is the following: suppose that the metainference

\[
\text{(1) From } \Gamma_1 \models A_1 \ldots \Gamma_n \models A_n \text{ we infer } \Gamma \models A
\]

is valid in classical logic, but not in ST. This metainference will be recovered in the next step of the hierarchy, which is ST_1. The same reasoning applies to ST_1: the classical metametainferences (inferences between metainferences) that are not valid in ST_1 are recovered in ST_2, and so on. This creates an infinite hierarchy of metainferences, and the system that reunites the logics ST_n (for n ∈ ω) is called STT_ω. The resulting theory of truth is called STT_ω. Now we will present some basic definitions that are taken from Pailos (2020a).

**Definition 4.1.** An inference on L is a pair (Γ, ∆), where Γ, ∆ ⊆ INF_0(L) (written Γ = ∆). INF_0(L) is the set of all inferences on L.

Note that Definition 4.1 talks about inferences between inferences between sets of formulas whereas we have defined inferences between sets of formulas and a formula. Fortunately, we can restate our former definitions as follows:

Γ = ∆ (in L) iff: if every γ of Γ is satisfied according to L, then so is some δ of ∆.

**Definition 4.2.** A metainference of (a finite) level n (for 1 ≤ n ≤ ω) is an ordered pair (Γ, ∆), where Γ, ∆ ⊆ INF_{n-1}(L) (written Γ =_n ∆). INF_n(L) is the set of all metainferences of level n-1 on L.

Before we continue, let us introduce a notion from matrix semantics and a terminology. As the reader can note, the matrix of MSC has only one set of designated values, whereas the matrices for ST and TS have two sets. As Chemla et al. (2017) calls, these sets are called standards. A standard is a subset of the set of values of the matrix of L that determines what means for a valuation v of SEM_L to confirm or to refute an inference. So, the matrices for ST and TS have two different standards, one for the premises and one for the conclusions. Second, the valuations that respect the truth tables of negation, conjunction and disjunction above presented are called Strong Kleene valuations (for short, SK-valuations).

**Definition 4.3.** A metainference Γ =_n ∆ is valid in a logic L if and only if, for every v of SEM_L, if v satisfies every γ of Γ according to L, then v satisfies some δ of ∆ according to L.

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3 Here we will omit the subscript on the relation ‘|=’ whenever the context is clear.
Definition 4.4. Let $L_1$ and $L_2$ be two logics characterized by SK-valuations. $L$ is a mixed system if its consequence relation $|=_{L}$ is defined as follows: $\Gamma |=_{L} \Delta$ if and only if it is not the case that $v$ satisfies all $\gamma$ of $\Gamma$ according to $L_1$ and not satisfies any $\delta$ of $\Delta$ according to $L_2$.

Given the basis ST of the hierarchy ST-ω, we will define the others ST-n that are mixed, in the sense of Definition 4.4. When $n = 1$, we have the mixed logic TSST, whose definition of consequence relation is stated as follows:

Definition 4.5. $\Gamma |=_{L} \Delta$ is valid in a logic TSST if and only if, for every $v$ of SEM-TSST if $v$ satisfies every $\gamma$ of $\Gamma$ according to TS, then $v$ satisfies some $\delta$ of $\Delta$ according to ST.

As Definition 4.5 highlights, the logic TSST is defined in such a way that it preserves the original intuition of ST because its consequence relation is defined “from the strict to tolerant.” As we can see, in a valid metainference of TSST, the premises receive the value 1 and the conclusion or 1 or $\frac{1}{2}$. As we said before, ST-1 ($= TSST$) recovers the valid inferences of ST.

The following example shows how this recovery happens. Consider again (Meta-explosion). That is, the following metametainference:

$$\{ \Gamma_1 \models 1 A, \Delta \}, \{ \Gamma_2 \models 1 \neg A, \Delta \} \models 1 \{ \Gamma_3 \models 1 B, \Delta \},$$

Where $\Gamma$ and $\Delta$ are inferences of ST, is not valid in TSST. That is, TSST is not closed under its metametainferences. So, Pailos (2020a) introduces an infinite hierarchy of metainferences where the logics that are on the higher levels recover the metainferences from the logics on the lower levels. In what follows, $L_1/L_2$ denotes a mixed system. Consider the following definitions:

Definition 4.6. For any metainferential consequence relation $L_1/L_2$, $\#L_1/L_2\# = L_2/L_1$.

Definition 4.7. The first step of the hierarchy is the following:

(i) ST-1 = TSST;

(ii) For every $2 \leq n < \omega$, the consequence relation of ST-n is defined as follows:

Let $\Gamma_1, ..., \Gamma_k$ and $\Delta$ be metainferences of level $n-1$. $\Gamma_1, ..., \Gamma_k |_{n} \Delta$ is valid if and only if every valuation $v$ that satisfies $\Gamma_1, ..., \Gamma_k$ according to $\#\Gamma_{n-1}$, also satisfies $\Delta$ according to $\Gamma_{n-1}$.

Now we present some collapse results between ST-n and classical logic.

Fact 4.8. For every level $n$ ($1 \leq n < \omega$), a metainference $\Gamma_1, ..., \Gamma_k |_{n} \Delta$ is valid in classical logic if and only if it is valid in ST-n.
However, as Pailos (2020a) observes, $\text{ST}_n$ is not closed under its metainferences of level $n + 1$. For example, there will be instances of Cut, Meta-explosion, Meta-modus ponens in the level $n + 1$ that are not valid in $\text{ST}_n$. This is expressed by the following fact:

**Fact 4.9.** For every level $n (1 \leq n < \omega)$, $\text{ST}_n$ invalidates infinitely many classical metainferences of level $n + 1$. (And its metainferences of level $n + 1$ are properly included in the $n + 1$ classical metavalidities.)

In order to overcome this limitation, Pailos (2020a) defines $\text{ST}_\omega$ as the union of the logics $\text{ST}_n$.

**Definition 4.10.** A metainference of level $n \Gamma \models_n \Delta$ is satisfied in $\text{ST}_\omega$ if and only if it is satisfied in some $\text{ST}_n$.

The logic $\text{ST}_\omega$ makes it possible to recover every classical metainference. Every classical metainference that is lost in some $\text{ST}_k$ is recovered in its successor $\text{ST}_{k+1}$. Then so it is recovered in $\text{ST}_\omega$.

Now, it is important to make some observation regarding the above construction. As Ferguson & Ramírez-Cámara (2022) observe, the objects of the inferences of the logics $\text{ST}_n$ are not formulas, but metainferences. Second, each logic of this hierarchy is treated as an instance. Given that there is no language that take these metainferences as primitive objects, the metainferences are instances as well as each logic in this hierarchy. This is by no means a limitation of Barrio et al.'s construction. Indeed, it can be seen as an advantage, because this allows generalizations that can be applied to other logics.

In what concerns the truth predicate, Pailos extends the language $L_{\text{ST}}$ with $\text{Tr}$, thus obtaining the theory $\text{STT}_\omega$. As he defines, this theory is supposed to contain a name forming operator that provides names $[A]$ for every sentence $A$. In the language of first-order logic, every sentence $A$ has a distinguished name that is represented by an individual constant, that we can denote by $c_A$ to make our presentation simpler. So, the individual constant $c_A$ stands for $[A]$ in the first order language of the theory $\text{STT}_\omega$. Thus, in what concerns the first-order models, the domain $D$ of the interpretation contains those distinguished names that are represented by these individual constants.

**Definition 4.11.** A model $v$ for $\text{STT}_\omega$ is a model for $\text{ST}_\omega$ that satisfies the following restriction: $v(A) = v(\text{Tr}([A]))$.

From this definition, it is clear that (Schema-Tr) is valid in $\text{STT}_\omega$. In what concerns the semantic paradoxes, we have the following situation: since that each $\text{ST}_k$ fails to validate the metarule of Cut in the level $k + 1$, the inconsistency caused by (Liar sentence) and (Curry

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4 Ferguson & Ramírez-Cámara (2022) propose a different approach to ST-hierarchy, by treating metainferences as objects in this hierarchy. That is, they define an extended language, call it $L'$, that contains terms for each metainference of $\text{ST}_\omega$. 

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* RHV, 2023, No 22, 85-103

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sentence) is avoided in STTω. Pailos (2020a) proves that is non-trivial. That is, it has a model that satisfies all the sentences of its sentences. As proved in Barrio & Bezerra (2023) show, STTω can be non-trivially extended with a validity predicate.

4.1. STTω, classicality and objections

The semantic theory STTω is a good candidate for the philosophical project of providing a semantically closed language, i.e. a language that can talk about its own semantic concepts. Moreover, as we showed in Section 4, the logic STω recovers classical metainferences in its hierarchy because there is always a STn that recovers the classical metainferences at issue. But it raises the following question: is STω classical logic? The answer is no. The reason is the following; even if STn always recovers classical metainferences at some point STk in the hierarchy, it fails in validating metainferences of level k + 1. On the other hand, the metainferential hierarchy based in classical logic is classical at every metainferential level. Here we have the following passage taken from Barrio et al. (2020) (hierarchy of classical and paraconsistent logic):

Remarkably, such a criterion allows to provide an answer to our initial question: what identifies CL as such? It is, in fact, the inferences it validates at every inferential level that are crucial to its identity and which allow to tell it apart not only from closely related systems like ST, but also from systems that are even more similar to it—like TS/ST or any of the recursively defined systems appearing in our hierarchy, for that matter. What identifies CL are all its valid inferences of every inferential level, and each of those systems differ with it at some point. This clearly explains why ST is not CL, and neither is TS/ST or any of the systems of our hierarchy. Subsequently, this criterion makes it also easy to tell the difference between the systems in the hierarchy themselves. (Barrio et al., 2020, p. 114)

Although STω is not classical logic, it recovers classical metainferences as most as possible. In comparison to other non-classical solutions to semantic paradoxes, STTω is a remarkably strong system. For this reason, we claim that STTω is an adequate solution to semantic paradoxes, for being classical as far as possible.

However, cut-free solutions based on STTω faces objections in the literature. For example, Scambler (2020), Golan (2022) and Porter (2023) argue that STω is not a logic because it is not closed under its own metainferences. As we showed above, STn is not closed under the metainferences of level n + 1. For this reason, they claim that it is not possible to reason with STω because it does not allow to connect inferences in order to make another inferences. As argued pointed by Barrio & Bezerra (2023), this failure of closure is not specific to STω, but to all the logics characterized by SK-valuations. While ST fails to be closed under its metainferences, TS has no valid inferences and no tautologies, the logic LP (Asenjo, 1969; Priest, 1979) and K3 (Kleene, 1938) have mismatches between inferences and tautologies:
LP has the same tautologies as classical logic, but fails to validate modus ponens; $K_i$ has no tautologies while it validates many classical inferences. That is, all non-classical solutions based on SK-valuation will present some kind of mismatch between its inferences, validities and metainferences.

Second, as we pointed before, the hierarchy $ST^\omega$ shows the way how we perform inferences using this logic. As Barrio et al. (2021) explains, the whole hierarchy formalizes reasoning according to the standard from strict to tolerant. That is, the inferences formalized by this logic should be understood from the perspective of mixed logics. So, $ST^\omega$ indeed connect metainferences when we read the premises, that are also metainferences, as strictly valid and the conclusion, another metainference, as tolerantly valid. It is according to this perspective that the inferences formalized by $ST^\omega$ should be understood.

Ferguson & Ramírez-Cámara (2020) also raise some criticisms concerning the construction of the hierarchy $ST^\omega$. According to them, the hierarchy $ST^\omega$ has a limited expressivity that, by its turn, does not allow to represent metainferences involving metainferences of distinct levels. So, they propose a language that considers metainferences as primitive objects as well as a new semantics for the hierarchy that is based on the conditional $\rightarrow$ of LP, that is definable as $\neg A \lor B$.

In order to respond Ferguson & Ramírez-Cámara’s criticisms, we note the following: even if Barrio and his collaborators’ proposal do not take metainferences as primitive objects in the language, Ferguson & Ramírez-Cámara can be properly embedded in the hierarchy $ST^\omega$, as defined above. The reason is the following: Barrio et al. (2020) and Roffe & Pailos (2021) offer a procedure of translating metainferences into formulas. Intuitively, this procedure works as follows: an inference of the form

$$\Gamma \models \Delta$$

is converted into a formula of the form:

$$(\gamma_1 \land \ldots \land \gamma_n) \rightarrow (\delta_1 \lor \ldots \lor \delta_n)$$

where $\gamma_1, \ldots, \gamma_n$ belong to $\Gamma$ and $\delta_1, \ldots, \delta_n$ belong to $\Delta$. This procedure can be applied to metainferences of higher levels. In the case of $ST$, its conditional has the same truth-table as in LP. For this reason, we claim that Ferguson & Ramírez-Cámara’s proposal can be reproduced in the original $ST^\omega$ as presented above. So, the construction of the hierarchy $ST^\omega$ enjoys expressivity enough to reproduce metainferences of distinct levels.

5. Conclusion

We argued that the theory $ST^\omega$ is an adequate non-classical theory of truth since it keeps classical reasoning intact as most as possible, and it is able to recover the classical metainferences with the aid of the logics in the hierarchy. Since the metarule of Cut of level $n + 1$ is not valid in $ST^\omega$, one cannot expect to reproduce stronger versions of liar and Curry
paradoxes in ST\(_\omega\). As showed by Barrio & Bezerra (2023), STT\(_\omega\) can be extended with a validity predicate that satisfies both (VP) and (VD) without trivializing the theory. So, the cut-free approach to semantic paradoxes is capable to block the paradoxical results mutilating classical logic the least possible.

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